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AUTHOR Buck, Judy Curran  
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## ABSTRACT

The typical path of instruction in high school algebra courses for the study of polynomial functions has been from linear functions, to quadratic functions, to polynomial functions of degree greater than two. This paper reports results of clinical interviews with an Algebra II student. The interviews were used to probe into the student's conceptual understanding of polynomial functions and the connections that the student made graphically and algebraically between the classes of polynomial functions. Results suggest that building polynomials from linear expressions may foster connections between the classes of polynomial functions and between the graphical and algebraic representations of these functions.

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# FOSTERING CONNECTIONS BETWEEN CLASSES OF POLYNOMIAL FUNCTIONS

Judy Curran Buck

University of New Hampshire  
Durham, NH 03824

and

Plymouth State College  
Plymouth, NH 03266  
jcurran@oz.plymouth.edu

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## **Introduction and Rationale**

The typical path of instruction in high school algebra courses for the study of polynomial functions has been from linear functions, to quadratic functions, to polynomial functions of degree greater than two. Furthermore, according to Philipp, Martin, and Richgels (1993), most existing algebra textbooks today present *graphs* in stand-alone sections or chapters instead of integrating them with corresponding algebraic procedures. In other words, graphs are treated as ends in themselves.

The *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 1989) highlight the importance of linking conceptual and procedural knowledge among the different topics in mathematics. In this way students are encouraged to see the patterns throughout mathematics, and to view mathematics as an integrated whole, rather than as a series of isolated topics. "The Standards" claim that "developing mathematics as an integrated whole also serves to increase the potential for retention and transfer of mathematical ideas." (p. 149)

While *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) advised mathematics educators of the need to promote connections in the mathematics curriculum, mathematics educators are also realizing the impact that technology is having in the classroom. Traditionally the emphasis has been on algebraic procedures for finding solutions to polynomial equations, but now, with the aid of technology, i.e. computers and graphing calculators, more emphasis is being placed on the graphical representation as a means of

accomplishing these same tasks. The increased emphasis on the graphical representation is by no means an attempt to replace the use of the algebraic representation, but rather a means by which the algebraic representation can be illuminated to promote student understanding.

With these changes, it is appropriate for research that investigates the connections that students' make between these classes of polynomial functions, graphically as well as algebraically.

This paper reports some of the results from an investigation into students conceptual understandings of polynomial functions. The focus here will be on results that pertain to the connections that students make graphically and algebraically between the various classes of polynomial functions as determined by their degree.

### **Specific Aims**

The current study was motivated by the following questions:

1. What connections do the students make between the classes of polynomial functions?
2. What evidence suggests that the students' conceptual understanding of the graphs of polynomial functions of degree greater than two connects to or builds upon their understandings of the graphs of linear and quadratic functions and their general knowledge of functions and graphs?
3. What cognitive factors can be identified that contribute to/inhibit the student from making the transition from the graphical representations of linear and quadratic functions to those of higher degree?
4. What influence does technology have in influencing/inhibiting the students from making connections between the classes of polynomial functions?

5. Do student observations suggest activities that would make the connections between classes of polynomial functions more salient?

### **Research Assumptions**

The following assumptions were held by the researcher in designing this study.

1. *Linear and quadratic functions are a foundation for further study in polynomial functions of higher degree.*

These two classes of polynomial functions can be thought of as the “building blocks” for all other classes of polynomial functions. Therefore, there are concepts that can and should be transferred from the study of the graphs of linear and quadratic functions to the study of the graphs of polynomial functions of degree greater than two.

2. *Connections should be made between all classes of polynomial functions as determined by their degree.*

Instruction should focus on regarding the sundry classes of polynomial functions as variations of one another rather than as unrelated entities.

### **Background: Algebra Reform And The Impact Of Technology**

The current interest in the graphical representation of functions has been spurred on by advances in hand-held technology that now make it possible to quickly generate graphs of functions, a task that was previously considered tedious. Before graphing technology was available to the classroom, the amount of time and effort needed to construct graphs outweighed the benefits of using the graphical representation. Consequently, there was a general dependence on algebraic manipulations to solve problems. Tasks such as finding the roots of a

polynomial function of degree greater than two had traditionally been a trial and error process using the Rational Root Theorem. Now, however, graphing technology used in conjunction with the Rational Root Theorem can ease the search for solutions. To summarize, though algebra has usually been regarded as a language of symbols, technology seems to be widening the view of what algebra is and how it should be taught.

The use of graphs could even increase the intrinsic interest of factoring, as students explore the connections between graphical and algebraic representations of functions. Graphing utilities can help students discover the importance of factoring and enhance their interest in it by greatly reducing the emphasis on an algorithmic, manipulative approach, while revealing the significant information available from the factored form of an expression. (Philipp et al., 1993, p. 268)

### **Conceptual Framework**

Prior research shows that what a student sees in a graphical representation depends on his/her existing knowledge about functions and graphs (Schoenfeld, Smith, & Arcavi, 1993; Larkin, 1987). Students see the graph through their current understanding of graphs of that type and through other experiences with graphs of that type. In other words, graphs are not transparent. Students don't always see what we want or expect them to see, i.e. they see what their conceptual understanding at that time prepares them to see.

The underlying framework for the proposed study is a constructivist position on conceptual change (Pintrich, Marx, & Boyle, 1993). This suggests that the process of conceptual change is affected by the active participation of the learner in the construction of his or her own knowledge and can be facilitated or hindered by the learner's beliefs and motivation, as well as the effects of contextual and situational factors.

Guided by the aforementioned research questions and this theoretical framework, the research was accomplished through classroom observations, clinical interviews, and teaching episodes.

### **Research Design**

The study took place within an upper level Algebra II class at a high school in northern New England. After videotaping the class instruction that dealt with polynomial functions, I focused my interest on three students who had volunteered to participate in the study. Interactions with these three students occurred in three phases:

1. Clinical interviews: Working within the framework of a constructivist position on conceptual change, it was appropriate to conduct clinical interviews to investigate each students understanding. These were used to explore the students' understanding of functions and to identify the students' attitudes and beliefs about mathematics in relation to his/her previous mathematics background.
2. "Teaching episodes" (Steffe, 1984): Following these interviews, an effort was made, through the use of teaching episodes, to enhance connections between classes of polynomial functions and to enrich the students' understandings of the graphs. This intentional intervention in the student's knowledge construction was made in the light of the observations made in the clinical interviews.
3. Evaluation of the teaching episodes: Student understanding of the graphs of polynomial functions was probed once again in clinical interviews using the same graphs that were used in the original clinical interviews.

## **Data Analysis**

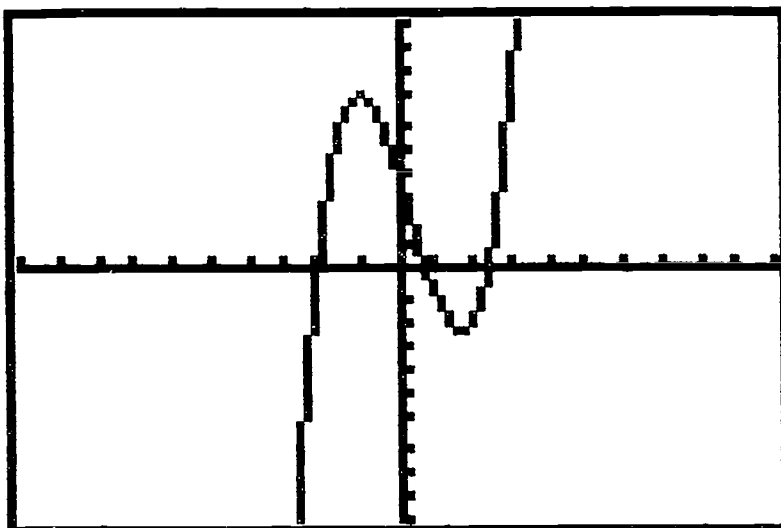
Guided by the aforementioned research questions, an analysis of these investigations in the clinical interviews and the teaching episodes was undertaken through the writing of case studies. Information was triangulated from various sources including field notes, classroom video tapes, a teacher interview, student quizzes and tests, student journal entries, and the video tapes of the interviews and teaching episodes. These case studies were written to describe in detail each student's developing understanding of the graphs of polynomial functions as understood by the researcher.

## **Results**

An analysis of the clinical interviews revealed that, due to the emphasis of instruction, students do not generally make connections between the classes of polynomial functions algebraically, though they do seem to try to make these connections graphically. This can be shown through excerpts from an interview with a student that I will call Mark, who was a junior in the Algebra II class. Using PC-Emulation Software (Texas Instruments, 1991), that corresponds to the handheld TI-81 (Texas Instruments, Inc.) graphing calculators that the students used in class, I was able to move a "cursor" along the polynomial function of degree three (Figure 1) with the help of a "mouse". Starting at the leftmost part of the graph, I moved up the curve and asked Mark about various locations on the graph. Stopping at the top of the "hill", I asked Mark if he thought that location on the graph had anything to do with the equation for this curve. Mark commented:

MARK: Um...You know, if it was like a parabola, you could plug in the ...oh, let me see...yeah, you could plug in the numbers in the... $a(x-h)^2 + k$ ...you could plug them in there and that would help you get the equation.





**Figure 1: PC-Emulation Software**

He was connecting the turning point on this graph with the vertex of a parabola determined by a polynomial of degree two. Continuing, I stopped the cursor on the y-axis, and asked Mark the same question. He reasoned:

MARK: That's probably...like in a parabola...that would be like what you add on at the end...if that was the vertex...or if it was a line you'd add that on.

He had connected the y-intercept on this graph to the y-intercepts of polynomials of both first and second degrees.

Though Mark made these connections between the classes of polynomial functions, he had not suggested that there was any connection between the x-intercepts of this curve and those in quadratic and linear functions. I found this interesting as a significant amount of class time had been spent in finding the values of the x-intercepts by using the Rational Root Theorem in conjunction with the graphing calculators. This strategy for finding the roots, however, held little, if any, resemblance for Mark to finding the roots of a quadratic or a linear equation. With quadratic equations, the classroom procedures for finding roots

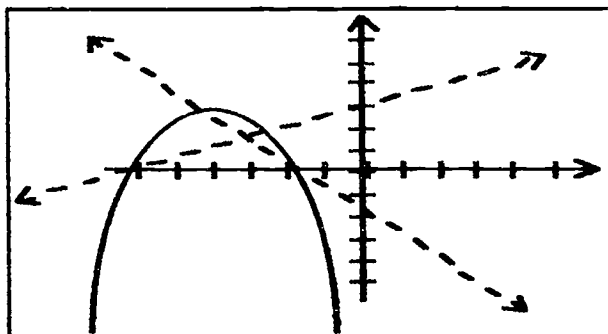
had emphasized the algebraic representation, by either factoring or using the quadratic formula. Graphically, the emphasis had been placed on the vertex and seldom on the roots. With linear equations, the x-intercept was simply the "place where  $y=0$ ", this point had never been called a "root". Both algebraically and graphically, the emphasis had been on the slope of the line and on the y-intercept. Mark did not see the connection that the real roots of any polynomial function are represented graphically by the points on the x-axis.

My hypothesis was that giving the students the opportunity to build polynomial functions by taking products of linear functions would make these connections more salient. Furthermore, the use of software that allowed the student to see the algebraic representation alongside the graphical representation would perhaps foster the formation of these connections.

Fortunately, the Educational Development Center in Newton, Massachusetts had already developed an ideal software tool to be utilized in the planned teaching episodes entitled *The Function Supposer: Explorations in Algebra* (Educational Development Center, 1990). One program option in this computer environment is the construction of polynomial functions by taking products of linear functions. The software allows the user to see the graphs of the linear function components as well as the resulting polynomial function on the same graph. The algebraic representation can be shown beside these graphs.

Using this software, tasks were created which would give the students the opportunity to construct polynomial functions by taking products of linear expressions. Mark would enter two linear functions and see both algebraically and graphically how taking the product of the linear expressions resulted in a quadratic function /parabola. An example is shown below (Figure 2) where Mark

typed in the functions  $f(x) = \frac{1}{2}x + 3$  and  $g(x) = -x - 2$ . Before the parabola was shown, the quadratic function was written algebraically as  $h(x) = (\frac{1}{2}x + 3)(-x - 2)$ . I asked Mark to predict the positioning of  $h(x)$  on the graph. Part of this conversation follows.



**Figure 2: Building quadratic functions**

JC: Can you...do you have any feel for what the graph of the product of those two would be?

MARK: Oh, it would be a parabola.

JC: How do you know that?

MARK: Cause it'd be...you'd have  $x^2$ .

JC: O.K., and do you know anything about the parabola?...anything at all?

MARK: Eeee...it would be...um...it would be wide....Cause you have  $\frac{1}{2}$  there so when you take that out you'd get  $x + \frac{1}{2}$  ...multiply that...multiply it by  $\frac{1}{2}$ ...um...I gotta do all the multiplication in my head...um...let's see...it'll open down.

JC: How do you know that?

MARK: Cause that's a  $-x$  and when you multiply that through...and I don't know...it's not going to be on the origin...I'm not sure where it would be...I'd need a pencil and paper to figure that out.

Mark was obviously not gaining information about the graph of the parabola from either the graphs of the linear expressions or the algebraic linear expressions themselves. Not until he multiplied the expressions out, did he have information about the graph. My sense was that he was multiplying these linear expressions together to get "standard form", and then converting standard form to the form  $y = (x-h)^2 + k$ , in order to get information about the graph.

Once we allowed the software to show the graph of the parabola, I asked Mark if the resultant parabola was what he expected.

MARK: Um...I think so. Is that wider?

JC: Well, yeah, probably.

MARK: Yeah, pretty much.

I probed Mark more about the parabola's position on the axes.

JC: Anything you note about where that parabola is?

MARK: Well, it moved on the x-axis...I didn't know it was going to do that, but I guess I kind of really figured it out when I did the problem...I figured it be on the y-axis, but...

JC: O.K., what about the x-axis?

MARK: It moved on it...it moved sideways.

JC: O.K., do you notice anything about where it...where it went?

MARK: Oh, its near the...the vertex is kind of close to the...intersection?

JC: Uh, huh...O.K. And how about where it crosses the x-axis?

MARK: [He looks at the graph.]...-2 and -6...is that right?...all right.

JC: How do you think...why do you think that?

MARK: No, that wouldn't be right...[He looks at the product of the linear expressions.] Yeah, cause if you take the negative out of that and then you have  $x + 2$  and that would make it...um...make the...ah...factor 2...-2...and you take the  $\frac{1}{2}$  out of that you get a 6...so you make it a -6.

JC: All right, and how does that correspond to the linear expression?

MARK: Oh...yeah, that's right where they cross...huh...huh...huh...

Mark was surprised that the parabola went through the x-axis at the same points as the lines did. It was not immediately obvious to him. His attention was drawn more to the global position of the graph relative to its basic position ( $y = x^2$ ), i.e. that it had moved left or right and was no longer centered on the y-axis. He had noted also that the vertex of the parabola was close to the intersection of the two lines. Not until I asked about how the parabola's x-intercepts corresponded to the x-intercepts of the lines, did he actually realize they were the same.

Mark did many more examples of this type and discovered that the parabola would always have the same x-intercepts as the lines. I then gave him some examples where he worked backwards, i.e. given the graph of a parabola, he

found two lines that he thought could be the "building blocks" of that parabola. We also extended this activity to polynomial functions of degree three that were composed of three linear functions. For each example, we talked about the relationship that these graphs had to their algebraic expressions, especially how the x-intercepts related to the *factors* of the expressions.

Mark wrote in his journal after this first teaching episode:

I found our last meeting very interesting. No one has ever explained that to me that way. I can really see why you get the parabola you get when you multiply lines together. Knowing why really helps, I thought I understood before but now I can really see it. Thank you, it was very interesting.

### Conclusions

The method of building polynomials from linear expressions used in the teaching episodes not only fostered connections between the classes of polynomial functions, but also fostered connections between the graphical and algebraic representations of these functions. In particular, the relation between the zeros of a polynomial  $f(x)$ , the roots of the equation  $f(x)=0$ , the factors of the polynomial, and the intercepts of the graph became more evident across classes.

It is probable that this research may hold implications for the classroom practice of mathematics educators as well as suggest modifications in the curriculum that will strengthen student's understanding of polynomial functions. Suggested improvements may result in the creation of modules for use in the classroom and also in teaching experiments that will serve to broaden the research in this area. To help pull together these seemingly fragmented topics, instruction on the graphs of polynomial functions of degree greater than two should build on the students previous knowledge of the graphs of linear and quadratic functions and other notions of function.

## References

- Larkin, J. H., Simon, H. A. (1987). Why a diagram (sometimes) is worth ten thousand words. Cognitive Science, 11, 65-99.
- National Council of Teachers of Mathematics (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: Author.
- Philipp, R. A., Martin, W. O., & Richgels, G. W. (1993). Curricular implications of graphical representations of functions. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), Integrating research on the graphical representation of functions. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Pintrich, P. R., Marx, R. W., & Boyle, R. A. (1993). Beyond cold conceptual change: The role of motivational beliefs and classroom contextual factors in the process of conceptual change. Review of Educational Research, 63(2), 167-199.
- Schoenfeld, A. H., Smith, J. P., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Eds.), Advances in instructional psychology. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Schwartz, J. & Yerushalmy, M., & Educational Development Center. (1988). *The function supposer: Explorations in algebra*. [Computer software]. Pleasantville, NY: Sunburst Communications.
- Steffe, L. P. (1984). The teaching experiment methodology in a constructivist research program. In M. Zweng (Eds.), Proceedings of the fourth international congress on mathematical education (pp. 469-471). Boston: Birkhauser.
- Texas Instruments, Inc. (1991). PC-81 Emulation Software [Computer software]. Lubbock, TX: Author.